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Boundary shock waves

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The possible occurrence of a viscous region near a surface from which there is rapid efflux of gas, accompanied by large heat transfer, is postulated and investigated theoretically. Such a viscous region, denoted as a boundary shock wave, may occur in the case of a large high-speed meteor entering the earth's atmosphere, when a very high rate of vaporization induces translational non-equilibrium. The conditions across a boundary shock wave and its structure are calculated from the appropriate macroscopic equations reduced to closed-form expressions under the restrictions of a perfect gas, flowing at constant total enthalpy.

1. Introduction

Recently attention has been drawn to the flow of gases at very high speed out of a solid or liquid surface. Examples of the mass-transfer process include: blowing of gas through a finely porous solid surface, sublimation, and vaporization from a liquid surface. This paper is concerned with, and postulates the occurrence of, a possible viscous gasdynamic phenomenon, denoted here as a *boundary shock wave*, under certain conditions in these flows. It appears not to have been considered previously.†

A boundary shock wave may be defined as a thin region of viscous flow adjacent to a surface from which gas is flowing at a very high rate (high Reynolds number based on characteristic dimensions of the injected flow) with significant heat conduction. In the limiting cases of one-dimensional flow normal to the surface, the viscous stress is purely viscous-compressive, as in a normal shock wave, with no shear present.

The postulation of this viscous boundary phenomenon arose in the consideration of what boundary conditions to apply in the calculation of the essentially inviscid bulk of the vapour (or injected gas) flow at high Reynolds number under conditions where heat conduction at the wall is expected. The rapid decay of the viscous stress and heat-conduction flux over a small distance would allow a rapid transition between an inviscid solution for the flow out of the wall and wall conditions (such as heat conduction) that would be otherwise incompatible with the inviscid-flow solution. The boundary shock wave is therefore a 'quick-transition region' belonging to the class of asymptotic phenomena discussed by Friedrichs

† The present paper is intended to clarify the boundary-shock concept introduced in *NASA TN D*-3195, January 1966, by the same author. That report may be consulted for details of the theoretical development.

(1955). It is like a boundary layer in that viscous effects are confined to the thin layer adjacent to the wall, but it is also like a viscous shock wave in that the flow is normal to the layer and the viscous stress is compressive rather than shearing.

The viscous effects characterizing the boundary shock wave may be most easily understood as effects of *translational* (and perhaps rotational) *non-equilibrium* of the molecular flow (to be discussed). The decay of viscous effects with distance from the surface corresponds to the relaxation of the translational non-equilibrium. In the limiting case of very large Reynolds number with steady flow normal to the surface, the macroscopic flow structure within the viscous region is governed essentially by the steady one-dimensional Navier–Stokes equations (*a*) if the gradients of velocity and temperature are not too large, and (*b*) if the flow is not turbulent (as may occur in the case of blowing through pores, for example). The same equations have been successfully used to study shock-wave structure. Professor Liepmann and his co-workers (Liepmann, Narasimha & Chahine 1962) have shown that the structure of most shock waves, especially in the downstream portion, is described very well by the Navier–Stokes equations. The structure of a boundary shock wave corresponds in many cases to a downstream portion of a simple shock-wave solution. (Note then that the boundary shock may be entirely subsonic.) Calculation of the flow through a boundary shock will therefore be based largely on the wealth of existing literature on shock waves (see especially the monographs by Hayes (1960) and Lighthill (1956) for theory and references).

A particular case where the possible occurrence of the boundary shock would be of interest is in the vapour flow in a certain class of meteoric fireballs entering the earth's atmosphere at very high speed. In this case, the translational non-equilibrium is induced by the high rate of vaporization, driven by a high rate of radiation absorption at the surface, with the accompanying necessarily large rate of *heat conduction* across the liquid layer that is required to raise it rapidly to vaporization temperature. An idealization of the meteor problem will be considered in a final calculation. (For pertinent discussion of large high-speed meteors, refer to Allen & James 1964.)

2. Possible occurrence of boundary shock waves

2.1. *General remarks*

In considering the calculation of flow over a body, such as a blunt body with continuous mass transfer out of the surface, one is interested in (*a*) possible viscous effects, and (*b*) possible regions where viscous effects can be neglected. The presence or absence of viscous effects in any region depends on two things: (*a*) some *source* of viscous effects, that is, a source of translational non-equilibrium, and (*b*) the magnitude of an appropriate local characteristic Reynolds number. A flow with small characteristic Reynolds number is all viscous if there is a source of viscosity or translational non-equilibrium. A flow with large characteristic Reynolds number is nearly all inviscid, except for possible very thin regions where the gradients, on which the viscous flux terms depend, are large. In those thin regions would be found some source of translational non-

equilibrium, such as shear at a boundary or at an interface, and such as the mixing of the molecules in a shock wave between two substantially different average states.

For this discussion, consider the example of steady gas flow out of a curved surface, with an opposing *external* flow and an interface between the two flows. Refer to figure 1, on which the directed lines are streamlines, b denotes the boundary, and L is the distance from the boundary to the stagnation point O on the interface. The boundary b is assumed to be where the mass transfer process

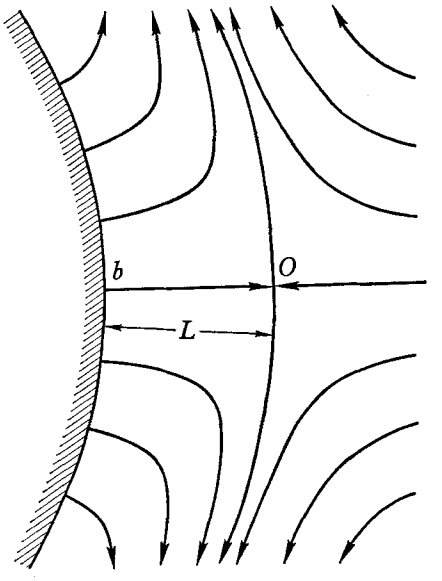

FIGURE 1. Flow with stagnation point.

is essentially completed and the flow is essentially normal to the surface. In the case of vaporization, b is a location where the phase change is essentially completed, that is, at a point where the molecules are essentially free of intermolecular forces except during isolated collisions. A characteristic Reynolds number for the injected flow is $Re_b \equiv L/l_v$, where $l_v \equiv \tilde{\mu}_b/\rho_b u_b$ is the 'viscous length' based on the longitudinal viscosity coefficient, $\tilde{\mu}$, mass density, ρ, and flow velocity, u, at b. As long as Re_b is *small*, the flow of the injected gas is entirely viscous. If also there is a *high* Reynolds number *external* flow, then the viscous-flowing injected gas is simply part of a *boundary layer* of the external flow. If the mass-transfer process is essentially an equilibrium process, then the source of viscosity in the injected flow is simply the shear at the interface. There is a large body of past and current literature on investigations of this aspect of mass transfer (with small Re_b).

2.2. *Conditions for occurrence of a boundary shock wave*

Consider now the case of very large Re_b. As Re_b first increases from the small value, the boundary layer is effectively blown off, leaving the viscous shearing effects confined to a thin region at the interface. As Re_b becomes very large, it is evident that the bulk of the injected flow must become essentially inviscid,

except within very thin regions surrounding any sources of translational non-equilibrium.

If there is no heat conduction (or source of translational non-equilibrium) at b in figure 1, then the flow at b for large Re_b is inviscid, and governed by the Euler equations of inviscid flow. If the influx were supersonic in that case, then there would have to be a detached shock wave somewhere within the injected flow in order for the flow to decelerate to subsonic and then zero velocity at the stagnation point.

If there *is* heat conduction or viscous stress at the boundary b, then the Euler equations are not uniformly valid up to the boundary, since they do not admit heat conduction or viscous stress. However, the Euler equations are just an approximation to the Navier–Stokes equations, which contain the higher derivatives that (*a*) *allow* for additional conditions on heat conduction and viscous stress to be satisfied at the wall; and (*b*) *allow* rapid variation in a very thin region to meet those conditions. Thus if there is significant heat conduction or viscous stress (translational non-equilibrium) at the boundary of the injected gas with large Re_b, that condition could *not* be satisfied by an inviscid flow, so there would have to be a thin viscous region of rapid transition near the body in order that the condition could be satisfied. That region of rapid variation of the flow properties, if it occurs, is denoted as a boundary shock wave. Conversely, for a boundary shock wave to occur in figure 1 for large Re_b, there would need to be a *source* of translational non-equilibrium at (or just to the left of) b. Apparently, therefore, the necessary and sufficient conditions for the occurrence of a boundary shock wave are: (*a*) large Re_b, *and* (*b*) significant heat conduction or viscous stress (significant translational non-equilibrium) at b.

The question of whether a boundary shock wave will occur is now seen to be equivalent to the question of whether there is significant viscous stress and/or heat conduction (i.e. significant translational non-equilibrium) in the gas at b for large Re_b. In each specific problem, in order to determine whether a boundary shock would occur, or, equivalently, whether the viscous stress τ and heat-conduction flux q_c at b are significantly different from zero, a completely determined system of equations must be satisfied in order to *determine* τ_b and q_{c_b} along with the other boundary parameters.

2.3. *Significance of the equilibrium solution*

In any given problem with large Re_b, it will be possible to find an 'equilibrium solution', that is, a solution with no boundary shock wave, by arbitrarily assuming $q_{c_b} = \tau_b = 0$ instead of applying an appropriate 'translational-non-equilibrium condition' (say, on the velocity, or temperature, or heat flux, etc.) at the boundary. Arbitrarily assuming τ and q_c to vanish identically would yield a determined set of equations if the appropriate translational non-equilibrium condition is omitted (to be seen in §4). Under some conditions this local translational equilibrium assumption would be justified, but not generally. When the vaporization rate is high, so that significant translational non-equilibrium is expected (to be discussed), one may include the translational non-equilibrium variables τ and q_c in the equations, use the phenomenological relations expressing

τ and q_c in terms of gradients of flow variables, use a roughly approximate translational non-equilibrium boundary condition on the velocity, and determine the effects of translational non-equilibrium which constitute the boundary shock wave.

2.4. *Translational non-equilibrium in very rapid vaporization*

It is well known that vaporization at low and moderate rates is, for all practical purposes, an equilibrium process, taking place reversibly at constant temperature and pressure. However, at sufficiently high vaporization rates, one must suspect possible translational non-equilibrium effects to become significant in the vaporization process. Those effects are described in this section.

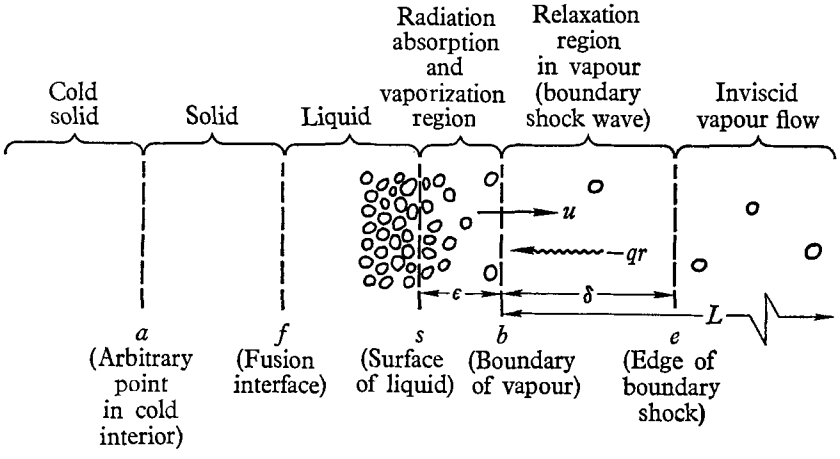

FIGURE 2. Diagram of vaporization.

2.4.1. *Rapid vaporization driven by external radiation.* Consider very rapid vaporization, driven by absorption of intense external radiation from the right in figure 2. Consider only a material that vaporizes from a liquid surface and has no chemical reactions after the vaporization. (In this idealization, as for a material of monatomic molecules, the vaporization is a purely physical process.) Because of the large radiative heat flux from the right and sharp temperature drop-off with distance into the liquid, only the liquid surface is at vaporization temperature. As a result, it is assumed that there is no bubbling or spraying of liquid drops. The latter is probably valid for a viscous liquid such as meteoric stone, but may not be for meteoric iron (see Öpik 1958). Assume the radiation is absorbed and the molten material vaporized between the 'surface' s and the 'boundary of the vapour', b. At b, the material has attained both the vapour density and the increased 'internal energy of vaporization', or the added potential energy of the individual molecules that have been separated from the constraints of the intermolecular forces. If the material at liquid density is highly opaque to the radiation and the vapour is relatively transparent to the radiation, then the distance from s to b within which most of the radiation is absorbed, would be extremely small, on the order of a few molecular diameters. Denote that distance by ϵ.

2.4.2. *Expected translational non-equilibrium.* Some of the absorbed-radiation

energy is used to vaporize the liquid molecules within ϵ by giving them sufficient additional translational energy to completely separate them from each other against the intermolecular forces that held them in the liquid state. Some of the energy must be *conducted* into the liquid by collisions in order to raise the temperature of the liquid so that it can be vaporized an instant later. The molecules absorbing the radiation within ϵ conduct the heat.

At small vaporization rates, the conduction heat flux is small, as evidenced by the fact that the process is then known to be nearly reversible (no significant entropy production). As the radiation flux increases, however, the conduction heat flux must increase to make the vaporization rate increase. As the conduction and vaporization rates increase, a condition should be reached where the vaporization can no longer be considered as an equilibrium process and is no longer reversible.

With a large conduction heat flux present as the molecules begin to separate into the vapour state, the translational non-equilibrium evidently would occur as a lag (on the average) in transferring the energy from the translational mode in the axial direction (normal to the surface) to the lateral degrees of freedom of the molecules. Denote the co-ordinate in the axial direction by x (distance to the right of b in figure 2), the lateral co-ordinates by y, z, the molecular velocity in the x-direction by $u+c_1$ (where u is the average velocity) and either of the lateral molecular velocity components by c_2. Conduction to the left is caused by the c_1 components. A number of collisions would be needed to get the same random energy distribution in the lateral modes as in the axial mode, so the above mentioned lag would occur in the accommodation of the lateral energy modes because of spaces between the molecules in ϵ. The lag makes $\overline{c_2^2} < \overline{c_1^2}$, where $\bar{(\ )}$ is the average, weighted by the velocity-distribution function, over all values of velocity (e.g. see Vincenti & Kruger 1965). The viscous stress τ is a measure of this lag, as can be seen from the one-dimensional-flow kinetic-theory expression for a perfect monatomic gas, $\tau = \frac{2}{3}\rho(\overline{c_2^2} - \overline{c_1^2})$, corresponding to which also is the heat-conduction flux $q_c = \frac{1}{2}\rho\overline{c^2 c_1}$. Because of the lag, τ is negative. If the temperature gradient and velocity gradient are not too large by the time the vapour state is achieved (at b), the viscous stress and heat-conduction flux can be represented there also by the linear relations

$$\tau = \tilde{\mu}\, du/dx, \quad q_c = -k\, dT/dx, \tag{2.1}$$

where

$$\tilde{\mu} = \lambda + 2\mu = \tfrac{4}{3}\mu + \tfrac{1}{3}\kappa, \tag{2.2}$$

and where μ is the shear-viscosity coefficient, λ is the second viscosity coefficient, κ is the bulk viscosity coefficient, and k is the coefficient of thermal conductivity. (2.1) can be regarded either as the phenomenological relations or as derived as the first translational non-equilibrium *corrections* to the local-equilibrium (Euler) approximation in the Navier–Stokes level of the Chapman–Enskog procedure in kinetic theory. Thus, if the accommodation lag exists at b, the velocity gradient there, from (2.1), is negative. The viscous stress τ, and its corresponding viscous dissipation of energy, are then seen to constitute a retarding effect opposing the acceleration in the phase change. From another point of view, a

negative value for τ is *compressive*, thus opposing or retarding the *expansion* in the phase change.

The lag that develops in the extremely thin region of the radiation absorption and phase change (i.e. within ϵ) must then relax, or equilibrate, with distance away from the source. That is, the viscous effects must decay. The distance within which the viscous effects decay is the thickness of the boundary shock wave. That viscous relaxation will be determined in §3 by solving the Navier–Stokes equations.

2.4.3. *Relationship between accommodation lag and density undershoot in phase change.* It will be noted that if the lag occurs, so that $\tau_b = (\tilde{\mu}\, du/dx)_b$ is negative, then the velocity must reach a maximum slightly before the end of the phase change, and so by conservation of mass, the density would at the same time have a minimum value. This possibly unexpected 'density undershoot' could be explained as follows. Let the velocity gradient during the phase change be separated into two contributions

$$du/dx = (du/dx)_{\text{phase}} + (du/dx)_{\text{visc}}, \tag{2.3}$$

where $(du/dx)_{\text{phase}}$ is the contribution representing the expansion process due to the molecules separating in the phase change and is the only essential contribution to du/dx if the vaporization rate is low or moderate. This quantity is always positive and has a large peak (maximum slope of u curve) near the middle of the phase change. When the above mentioned lag occurs, $(du/dx)_{\text{visc}}$ represents the small *negative* contribution due to the retarding effect of that lag. It is expected that $(du/dx)_{\text{visc}}$ would change from zero to a small negative peak value, then begin to relax near the end of the phase change. At b, $(du/dx)_{\text{phase}}$ is zero and

$$(du/dx)_{\text{visc}} = (\tau/\tilde{\mu})_b < 0.$$

By plotting qualitative curves for $(du/dx)_{\text{phase}}$ and $(du/dx)_{\text{visc}}$ and graphically adding them together (see figure 3 in Martin 1966) to get du/dx, one will see that du/dx would first have a large positive peak, then go through zero to small negative values, with a minimum point before b is reached. Where du/dx goes through zero, u is at a maximum point and the density ρ is at a minimum point, just before the phase change is completed, thus causing the density undershoot in the phase change.

3. General theory of the plane laminar boundary shock wave in a perfect gas with $\widetilde{Pr} = 1$

3.1. *Asymptotic description and one-dimensional flow*

In the development of the boundary-shock-wave theory, two viewpoints may be taken. One may consider either a purely one-dimensional flow out of a plane surface of infinite extent at a value of Re_b based on an arbitrary distance from the surface, or one may consider a configuration such as sketched in figure 1 in the limit as $Re_b \to \infty$ (with normal injection), which turns out to be equivalent, locally, to the former viewpoint.

For the purely one-dimensional flow case, the outer boundary conditions for the viscous region are simply the relaxation conditions:

$$x \to \infty\colon\ \tau \to 0, \quad q_c \to 0. \tag{3.1}$$

From the other viewpoint: if Re_b is very large in figure 1, the viscous region at the body, if it occurs, must be very thin. The effects of curvature are then higher order and can be neglected. Thus in the limit as $Re_b \to \infty$, the wall can be considered to be plane, locally, with one-dimensional flow normal to the wall. Also in this limit, the slopes of the variables outside the thin region are higher order, so the limit process requires the variables (e.g. velocity, temperature, pressure) to approach asymptotically to the values just outside the region (u_e, T_e, p_e, etc.). Since these values are unknown, the asymptotic first-order outer boundary conditions can be written simply as

$$x \to \infty\colon\ du/dx \to 0, \quad dT/dx \to 0, \tag{3.2}$$

which are equivalent to conditions (3.1), with (2.1).

For either of these two viewpoints, the one-dimensional configuration sketched in figure 2 is appropriate in the case of rapid vapour ablation. In the case of blowing through a finely porous surface with a strong heat source present, if the flow is not turbulent, b is a location where the velocity vector is essentially normal to the surface.

The integrated forms of the exact equations for conservation of mass, momentum, and energy in one-dimensional steady flow in a non-accelerating co-ordinate system are

$$\rho u = m, \tag{3.3}$$

$$\rho u^2 + f = C_1, \tag{3.4}$$

$$\rho u(e + \tfrac{1}{2}u^2) + q - uf = C_2, \tag{3.5}$$

where m, C_1 and C_2 are constants of integration, ρ is the mass density, f is the sum of the 'surface forces' per unit area in the x-direction on an element of mass, e is the internal energy per unit mass, and q is the heat flux in the x-direction, positive to the right. These equations apply between any two points in figure 2. Therefore the general laminar theory of the boundary shock wave (essentially the region from b to e in figure 2) can be worked out in terms of 'boundary parameters' (conditions at b), and hence independently of the mass-transfer process.

With subscript b denoting the value of any variable at b, one may write by definition

$$x = x_b = 0\colon\ u = u_b, \quad T = T_b, \quad q_c = q_{c_b}. \tag{3.6}$$

In any given application these boundary parameters would be evaluated by solving a determined set of equations. An example will be considered in § 4.

For the calculation of the gas-flow region from b to e,

$$f = -p + \tau, \quad q = q_c + q_r, \tag{3.7}$$

where p is the thermodynamic pressure, τ and q_c are given by (2.1), and q_r is the radiative heat flux. For simplicity, radiation absorption and emission in this region are neglected, so that q_r is assumed constant from b to e.

It is convenient to make use of the specific enthalpy,

$$h = e + p/\rho, \tag{3.8}$$

and to consider a thermally and calorically perfect gas, for which the equations of state are

$$p = \rho RT, \quad h = c_p T, \quad e = c_v T, \tag{3.9}$$

where R, c_p, and c_v are constants related by

$$c_p/c_v = \gamma, \quad c_p - c_v = R. \tag{3.10}$$

It is also convenient to define the parameters

$$\widetilde{Pr} \equiv \tilde{\mu} c_p/k, \quad M_b \equiv u_b(\gamma RT_b)^{-\frac{1}{2}}, \quad C_{h_c} \equiv -q_{c_b}(\tfrac{1}{2}\rho_b u_b^3)^{-1}, \tag{3.11}$$

and the variables

$$\xi \equiv \rho_b u_b \int_0^x (1/\tilde{\mu})\,dx, \quad \bar{u} \equiv u/u_b, \quad \bar{T} \equiv T/T_b, \tag{3.12}$$

$$\theta \equiv u_b^{-2}[c_p T + \tfrac{1}{2}u^2 - (c_p T_b + \tfrac{1}{2}u_b^2)], \tag{3.13}$$

in terms of which (3.3)–(3.5), with conditions (3.2) and (3.6), are to be solved in the following sections. The special case $\widetilde{Pr} = 1$ will be considered. In general, an assumption of $\widetilde{Pr} = 1$ should not be over-restrictive since $\widetilde{Pr} \approx 1$ for most real gases, as pointed out by Liepmann, Narasimha & Chahine (1962).

3.2. *Structure of a plane boundary shock for* $\widetilde{Pr} = 1$

Use of $\widetilde{Pr} = 1$ yields

$$d\theta/d\xi = (\rho_b u_b^3)^{-1}(-q_c + \tau u), \tag{3.14}$$

so that the energy equation (3.5) with use of (3.3), becomes simply

$$d\theta/d\xi - \theta = (d\theta/d\xi)_b, \tag{3.15}$$

with the conditions

$$\xi = 0, \quad \theta = 0; \quad \xi \to \infty, \quad d\theta/d\xi \to 0. \tag{3.16}$$

The only possible solution is

$$\theta = 0 = d\theta/d\xi, \tag{3.17}$$

with the consequent results from (3.13) and (3.14)

$$\text{for} \quad \widetilde{Pr} = 1\text{:} \quad q_c = \tau u, \tag{3.18}$$

$$c_p T + \tfrac{1}{2}u^2 = \text{constant}, \tag{3.19a}$$

or

$$\bar{T} = 1 + \tfrac{1}{2}(\gamma - 1) M_b^2 (1 - \bar{u}^2). \tag{3.19b}$$

This is the 'constant-total-enthalpy solution', and (3.18) and (3.19) are the same results as are well known for a simple shock wave.

The momentum equation (3.4) with use also of (3.3), (3.9) and the result (3.19*b*), becomes

$$d\bar{u}/d\xi = a\bar{u} - b + c/\bar{u}, \tag{3.20}$$

with the conditions

$$\xi = 0, \quad \bar{u} = 1; \quad \xi \to \infty, \quad d\bar{u}/d\xi \to 0, \tag{3.21}$$

where $$a = \frac{\gamma+1}{2\gamma}, \quad b = 1+\frac{1}{\gamma M_b^2}+\tfrac{1}{2}C_{h_c}, \quad c = \frac{\gamma-1}{2\gamma}+\frac{1}{\gamma M_b^2}. \tag{3.22}$$

The solution for $d^2 > 0$, where

$$d = (b^2-4ac)^{\frac{1}{2}} \tag{3.23}$$

is $$e^{-2a\xi} = \left(\frac{a-b+c}{a\bar{u}^2-b\bar{u}+c}\right)\left(\frac{2a\bar{u}-b+d}{2a\bar{u}-b-d}\cdot\frac{2a-b-d}{2a-b+d}\right)^{b/d}, \tag{3.24}$$

from which, given γ, M_b and C_{h_c} one can calculate $\bar{u}$ versus ξ. Then the variation of the other variables with ξ is obtained by using (3.19*b*) for $\bar{T}$ and

$$\bar{\rho} \equiv \rho/\rho_b = 1/\bar{u}; \quad \bar{p} \equiv p/p_b = \bar{\rho}\bar{T}. \tag{3.25}$$

Results for a typical case are shown in figure 3.

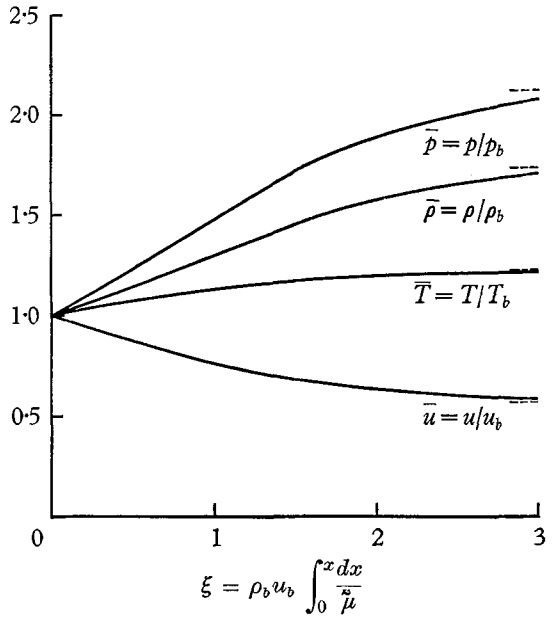

FIGURE 3. Typical boundary-shock structure ($\gamma = \frac{5}{3}$, $\widetilde{Pr} = 1$, $M_b = 1$, $C_{h_c} = 0{\cdot}5$).

3.3. *Conditions across a boundary shock for* $\widetilde{Pr} = 1$

With the result (3.18), one can obtain useful explicit expressions relating the conditions across a boundary shock wave (i.e. at b and e) for $\widetilde{Pr} = 1$ without further consideration of the detailed structure for each case.

From (3.3)–(3.5) can be obtained an expression for a boundary shock wave that is analogous to the Rankine–Hugoniot relation for a shock wave (see Liepmann & Roshko 1957, p. 64)

$$\frac{\rho_e}{\rho_b} = \left(1+\frac{\gamma+1}{\gamma-1}\frac{p_e}{p_b}-\frac{\tau_b}{p_b}\right)\Big/\left(\frac{\gamma+1}{\gamma-1}+\frac{p_e}{p_b}+\frac{\tau_b}{p_b}\right), \tag{3.26}$$

where $$-\tau_b/p_b = \tfrac{1}{2}\gamma M_b^2 C_{h_c}.$$

In a manner similar to the development of the Prandtl relation for a normal shock wave (see Liepmann & Roshko 1957, p. 57), one can also obtain for a boundary shock

$$u_e u_b = a^{*2} + \frac{2\gamma}{\gamma+1}\left(\frac{\tau_b}{\rho_e - \rho_b}\right), \tag{3.27}$$

where a^* is defined by

$$\frac{u_b^2}{2} + \frac{\gamma R T_b}{\gamma - 1} = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right) a^{*2}. \tag{3.28}$$

As $\tau_b \to 0$ (with $\rho_e/\rho_b \neq 1$), (3.27) reduces to the Prandtl relation, which determines that flow through a normal shock wave must go from either supersonic to subsonic or vice versa (see Liepmann & Roshko 1957, p. 57). For a boundary shock, with $\tau_b \neq 0$, such a restriction is not imposed, and both u_e and u_b may be subsonic.

Expressions for the 'jump conditions' (ρ_e/ρ_b, T_e/T_b, etc.) in terms of γ, M_b and C_{h_c} are easily obtained, for example, starting with (3.24), letting $\xi \to \infty$, $\bar{u} \sim \bar{u}_e$

$$\rho_b/\rho_e = u_e/u_b = (b/2a) - [(b/2a)^2 - (c/a)]^{\frac{1}{2}}. \tag{3.29}$$

Then from the energy and momentum equations

$$T_e/T_b = 1 + \tfrac{1}{2}(\gamma-1) M_b^2 [1 - (\rho_b/\rho_e)^2], \tag{3.30}$$

$$p_e/p_b = 1 + \gamma M_b^2 [1 - \rho_b/\rho_e + \tfrac{1}{2} C_{h_c}], \tag{3.31}$$

and

$$M_e/M_b = (u_e/u_b)(T_e/T_b)^{-\frac{1}{2}}. \tag{3.32}$$

Expressions in terms of γ, M_b and p_e/p_b will also be useful. From (3.31), (3.29) and (3.22) one can obtain

$$\rho_b/\rho_e = -B' + [(B')^2 + A']^{\frac{1}{2}}, \tag{3.33a}$$

where

$$A' = 1 + 2/(\gamma-1) M_b^2, \quad B' = (p_e/p_b)/(\gamma-1) M_b^2. \tag{3.33b}$$

Calculation of C_{h_c}, T_e/T_b, and M_e then follows from (3.31), (3.30) and (3.32).

Similarly, expressions may be found in terms of given γ, M_b and M_e by first eliminating T_e/T_b from (3.30) and (3.32) to obtain

$$\left(\frac{\rho_b}{\rho_e}\right)^2 = \frac{M_e^2}{M_b^2}\left(\frac{1 + \frac{1}{2}(\gamma-1) M_b^2}{1 + \frac{1}{2}(\gamma-1) M_e^2}\right). \tag{3.34}$$

Calculation of T_e/T_b, p_e/p_b and C_{h_c} then follows respectively from (3.30), from

$$p_e/p_b = (\rho_b/\rho_e) M_b^2/M_e^2, \tag{3.35}$$

and from (3.31).

With any of the above formulations, one can calculate the total change in specific entropy across the boundary shock as

$$\Delta s \equiv s_e - s_b = R(\gamma-1)^{-1} \ln[(p_e/p_b)(\rho_b/\rho_e)^\gamma], \tag{3.36}$$

of which the contribution due to *transport* of entropy is

$$\Delta_e s = -q_{c_b}/\rho_b u_b T_b = R\gamma M_b^2 \tfrac{1}{2} C_{h_c}, \tag{3.37}$$

and of which the remaining part is the entropy production $\Delta_i s = \Delta s - \Delta_e s$.

Example curves of ρ_e/ρ_b versus M_b are shown plotted in figure 4 for $\gamma = \frac{5}{3}$, $\widetilde{Pr} = 1$, and for several values of C_{h_c} (see Martin 1966 for further details and

results). For the limiting case $C_{h_c} = 0$, there is no boundary shock. If the efflux is subsonic ($M_b < 1$) and $C_{h_c} \to 0$, the boundary shock simply vanishes. If $M_b > 1$ and $C_{h_c} \to 0$, the boundary shock becomes a detached simple shock wave (cf. §2.2).

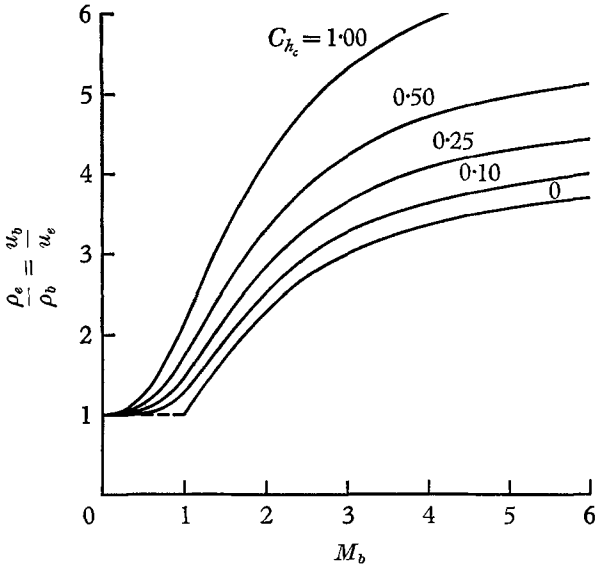

FIGURE 4. Density ratio and velocity ratio across a boundary shock ($\gamma = \frac{5}{3}$, $\widetilde{Pr} = 1$).

4. Approximate special application of the general theory to rapid vapour ablation

The purpose of this section is to apply the boundary-shock-wave theory to an approximate calculation of rapid vapour ablation of a material having the properties of a stone meteor entering the earth's atmosphere under conditions for which the boundary shock wave might be expected to occur. In this application, the parameters in the boundary-shock-wave theory will be evaluated by a determined set of equations, including a 'non-equilibrium condition' on the mass flux that allows non-trivial evaluation of the translational non-equilibrium variables, τ and q_c, as discussed in §§2.2, 2.3 and 2.4.

For simplicity the idealizations described in §2.4.1 and §3 are used. The thickness of the region from a to e in figure 2 is assumed to be small in comparison to the body curvature (which is easily justified after the calculation is made), so the first-order approximation of one-dimensional flow is valid. The applicability of the properties of a perfect gas in the vapour from a stone meteor will be discussed. The assumption of steady flow is here one of quasi-steady flow, for which time derivatives are neglected in comparison to space derivatives. The ablating surface is considered to be receding back into the liquid material with constant (locally in time) speed u_s, so that in figure 2 the surface $x = x_b = 0$ is stationary and the wall material (to the left of s) is moving toward the surface with constant velocity u_s.

4.1. *Equations for the solid and liquid*

For the solid and liquid regions in figure 2, an additional equation of state: ρ = constant is added to the system of (3.3)–(3.5), which apply to the material regardless of phase. The results are then

$$\rho_a = \rho_s, \quad u_a = u_s, \quad f_a = f_s, \tag{4.1}$$

$$\rho_s u_s (e_s - e_a) = q_{c_a} - q_{c_s}; \tag{4.2}$$

where

$$e_s - e_a = c_{\mathrm{sol}}(T_f - T_a) + L_f + c_{\mathrm{liq}}(T_s - T_f); \tag{4.3}$$

where subscripts a, f and s denote values at the respective locations: arbitrary point in the cold solid state (x_a), fusion interface (x_f), and surface just left of the radiation-absorption and vaporization region (x_s); and where c_{sol}, c_{liq}, and L_f are the assumed-constant specific heats in the solid and liquid and the latent heat of fusion at temperature T_f. The value of k is appropriate to the local state. In the cold interior of the wall, $T = T_a$ = constant, so the term $q_{c_a} = -(k dT/dx)_a$ in (4.2) is zero.

4.2. *Equations for the phase change*

If the flow equations were to be solved for the structure of the vaporization region ϵ (from s to b in figure 2), one would need, in addition to the conservation equations (3.3)–(3.5): (*a*) a thermal equation of state applying during the entire process, (*b*) an expression representing the internal energy during the process, (*c*) appropriate relations for τ and q_c during the process, and (*d*) a relation for the variation of q_r during the radiation absorption. Since, in this calculation, the structure of the region ϵ is not solved, but the equations are to be used to relate conditions at s and b, the required relations noted above must be replaced by appropriate conditions across ϵ, for the overall process, to make the system determined.

The conservation equations (3.3)–(3.5), with (3.7), evaluated between s and b, give approximately

$$-q_{r_b} = \rho_b u_b (e_b - e_s + p_b/\rho_b + \tfrac{1}{2}u_b^2) - q_{c_s} + q_{c_b} - \tau_b u_b, \tag{4.4}$$

where $1 - \rho_b/\rho_s \approx 1$ has been used, and in which τ and q_r are assumed to be zero at s.

The internal energy change across ϵ is assumed to be approximately

$$e_b - e_s = L_v - RT_b \tag{4.5}$$

for vaporization either at low rates or at high rates, where L_v is the latent heat of vaporization for low rates. (Thus, the internal energy of vaporization at a given temperature is assumed not to depend on the rate of vaporization.†) It is also then appropriate to assume $T_b \approx T_s$.†

An appropriate equation of state for the overall process of the phase change at low and moderate rates (reversible process) is provided by Clapeyron's equation

$$p_b/p_{\mathrm{ref}} = \exp\{(L_v/R)(1/T_{\mathrm{ref}} - 1/T_b)\}. \tag{4.6a}$$

Although Clapeyron's equation is derived assuming the phase change takes place reversibly at constant temperature and pressure, it appears to be qualitatively

† See appendix B of report: Martin (1966).

useful at very high vaporization rates as well, even if the pressure varies appreciably during the process because of the development of viscous stress. This may be partly due to the exponential (rapidly varying) dependence of p_b on T_b. Further, a proposed intuitive modification of Clapeyron's equation to account for significant viscous stress

$$(p_b - \tau_b)/p_{\text{ref}} = \exp\{(L_v/R)\,(1/T_{\text{ref}} - 1/T_b)\}, \tag{4.6b}$$

does not yield significantly different results in the final calculations (to be seen), so use of either (4.6*a*) or (4.6*b*) is assumed to be approximately valid. A basis for the proposition of (4.6*b*) is the following:† Clapeyron's equation (4.6*a*) gives the state of the material at which vaporization occurs at low rates, essentially in an equilibrium process. Then, for the translational non-equilibrium case, in which $p_b \neq p_s$, but in which $p_b - \tau_b \approx p_s$ (from the momentum equation, with $\rho u^2 \ll p - \tau$), one might expect the state of the material *at* s, where $\tau = 0$ and where the vaporization process begins, to be given by the same equation

$$p_s/p_{\text{ref}} = \exp\{(L_v/R)\,(1/T_{\text{ref}} - 1/T_s)\}.$$

Then with the assumptions $p_b - \tau_b \approx p_s$ and $T_b \approx T_s$, one obtains (4.6*b*).

A relation used by Öpik (1958, p. 24)

$$(\rho u)_b = \tfrac{1}{2}\rho_b(\bar{v}_x)_b \tag{4.7a}$$

(where ρ_b is the vapour density at saturation pressure and $\frac{1}{2}\bar{v}_x$ is the average component of molecular velocity in the $+x$-direction), appears to be an appropriate translational non-equilibrium condition, for vaporization at sufficiently high rates, to use for making the system of equations containing the translational non-equilibrium variables, τ and q_c, determined. It yields a mass flux with the same value as if the velocity distribution function were a one-sided Maxwellian distribution at the surface. It is taken as a reasonable gross approximation to describe the translational non-equilibrium of the rapid vaporization process, but the extent of its validity is not known. With evaluation of $\frac{1}{2}\bar{v}_x$ from kinetic theory and with the perfect gas equation of state for ρ at b, (4.7*a*) may be written

$$(\rho u)_b = \rho_b(\tfrac{1}{2}\bar{v}_x)_b = (p_b/RT_b)\,(RT_b/2\pi)^{\frac{1}{2}} = (2\pi R)^{-\frac{1}{2}} p_b T_b^{-\frac{1}{2}}, \tag{4.7b}$$

which is equivalent to another form used by Öpik (1958, p. 161). However, if (4.7*b*) is substituted into

$$M_b \equiv u_b(\gamma RT_b)^{-\frac{1}{2}} = (\rho u)_b\,(\gamma RT_b)^{-\frac{1}{2}}(p_b^{-1}RT_b),$$

a simpler form of the relation (4.7*b*) is obtained as

$$M_b = (2\pi\gamma)^{-\frac{1}{2}}. \tag{4.7c}$$

4.3. *Solution of the equations*

The equations for the regions $x < x_s$, x_s to x_b, and $x > x_b = 0$ are given respectively in §§4.1, 4.2 and 3.3 for $\widetilde{Pr} = 1$ in the vapour. Together they constitute a determined system if q_{r_b} and p_e are specified or obtained from an approximate solution

† The same result was obtained by a slightly different approach in appendix B of the report: Martin (1966).

of flow over a blunt body with large efflux from the surface. The equations are solved as follows.

Combination of (4.2) and (4.4), with use of (3.18) for $\widetilde{Pr} = 1$, gives

$$-q_{r_b} = \rho_b u_b(\zeta_{ab} + \tfrac{1}{2}u_b^2), \tag{4.8}$$

where, from (4.3) and (4.5),

$$\zeta_{ab} \equiv RT_b + e_b - e_a \tag{4.9a}$$

$$= c_{\text{sol}}(T_f - T_a) + L_f + c_{\text{liq}}(T_b - T_a) + L_v \tag{4.9b}$$

is the heat that must be transferred to a unit mass to raise it from the cold state at temperature T_a to the vapour state at temperature T_b. With use of $p_b = \rho_b RT_b$ and $u_b = M_b(\gamma RT_b)^{\frac{1}{2}}$, (4.8) may be written

$$-q_{r_b} = p_b\gamma^{\frac{1}{2}}M_b(RT_b)^{\frac{1}{2}}(\zeta_{ab}/RT_b + \tfrac{1}{2}\gamma M_b^2), \tag{4.10}$$

where $\gamma^{\frac{1}{2}}M_b$ is given by (4.7*c*), p_b is given by either of (4.6), and ζ_{ab}, as a function of T_b, is given by (4.9*b*). This essentially gives q_{r_b} as a function of T_b.

For meteoritic stone, Öpik (1958, p. 160) gives values for the mean atomic weight as 23 and the mean molecular weight of vapours as 50 g/mole. Hence $R = 1{\cdot}66 \times 10^6$ cm^2/sec^2 °K, and, since the mean molecular weight is about twice the mean atomic weight, we assume the gas mixture can be approximated as a perfect diatomic gas with $\gamma = \frac{7}{5}$. The reference values in (4.6*a*) or (4.6*b*) can be taken at the mean boiling point, given by Öpik (1958, p. 161) as 2960 °K (at $1{\cdot}013 \times 10^6$ dynes/cm^2). The values of quantities needed in (4.9*b*) are also given by Öpik (1958, pp. 61, 161):

$$L_f = 2{\cdot}65 \times 10^9 \text{ erg/g}, \qquad L_v = 6{\cdot}05 \times 10^{10} \text{ erg/g},$$
$$c_{\text{sol}} = 8{\cdot}95 \times 10^6 \text{ erg/g °K}, \qquad c_{\text{liq}} = 1{\cdot}1 \times 10^7 \text{ erg/g °K},$$
$$T_a = 200 \text{ °K}, \qquad T_f = 1800 \text{ °K}.$$

If (4.6*a*) is used, the procedure is now:

(*a*) specify q_{r_b} and p_e;

(*b*) choose trial values of T_b and iterate until (4.10), combined with (4.9*b*) and (4.6*a*), is satisfied; then T_b and p_b are known;

(*c*) compute the ratio p_e/p_b;

(*d*) find ρ_e/ρ_b from (3.33), T_e/T_b from (3.30), C_{h_e} from (3.31), and M_e from (3.32);

(*e*) find $\rho_b = p_b/RT_b$ and $u_b = M_b(\gamma RT_b)^{\frac{1}{2}}$.

Representative values of $-q_{r_b}$ and p_e used are, respectively, 17,300 watts/cm^2 and $7{\cdot}55 \times 10^5$ dynes/cm^2, found from a simplified approximate flow calculation for a spherical body of radius $r_b = 4{\cdot}62$ m, moving at 15 km/sec at an altitude of 60 km in the earth's atmosphere. The calculations outlined above lead to: $M_b = 0{\cdot}333$, $T_b = 2720$ °K, $p_b = 3{\cdot}345 \times 10^5$ dynes/cm^2, $\rho_b = 7{\cdot}41 \times 10^{-5}$ g/cm^3, $u_b = 2{\cdot}65 \times 10^4$ cm/sec, $p_e/p_b = 2{\cdot}257$, $\rho_e/\rho_b = 2{\cdot}215$, $M_e = 0{\cdot}149$, and $C_{h_e} = 13{\cdot}772$.

If instead the modified vaporization-state equation (4.6*b*) is used, the procedure is slightly modified. A calculation yields the results: $-q_{r_b} = 17{,}120$ watts/cm^2, $r_b = 4{\cdot}58$ m, $T_b = 2880$ °K, $\rho_b = 7{\cdot}00 \times 10^{-5}$ g/cm^3, and $u_b = 2{\cdot}73 \times 10^4$ cm/sec, with the remaining corresponding values the same as above. Thus it is seen that use of either (4.6*a*) or (4.6*b*) gives nearly the same results.

At this point one can see that the system of equations would be undetermined without a translational non-equilibrium condition such as (4.7*c*) unless one

arbitrarily specified $\tau_b = q_{c_b} = 0$ (or an equivalent specification such as $p_e = p_b$), in order to make the equations determined. That arbitrary assumption is not justified for the general case, as discussed previously. It seems unlikely that use of any other *independent* relation in place of (4.7*c*) to make the equations determined could make τ and q_c identically zero for all cases.

5. Summary and concluding remarks

An attempt has been made to present an introduction to a possible phenomenon defined herein as a boundary shock wave, a thin layer of gas through which viscous effects decay in very high-speed mass transfer accompanied by large heat transfer. Reasons were discussed for expecting occurrence of the viscous effects. The general theory of steady laminar flow through a plane boundary shock was developed. The theory was then applied to rapid vapour ablation from a body with flight conditions appropriate to a possible meteoric fireball in the earth's atmosphere, in order to completely determine the parameters in the theory for a particular case.

Although the actual occurrence of a boundary shock wave may not be unquestionably predicted, the following conclusions can be drawn from the discussions and theoretical developments in this paper:

(*a*) the necessary and sufficient conditions for a boundary shock to occur are: large efflux Reynolds number, *and* significant heat conduction or viscous stress (translational non-equilibrium) at the boundary;

(*b*) if the idealizations made are valid, and in particular, if the translational non-equilibrium condition on the mass flux used is approximately valid, then the theory and calculations will have predicted the occurrence of a boundary shock wave;

(*c*) similarly, if the non-equilibrium mass-flux relation is replaced by any other appropriate independent relation and yields significant non-trivial values for the non-equilibrium variables (viscous stress and heat flux) at the boundary, then the occurrence of the boundary shock would be predicted;

(*d*) it seems unlikely that any appropriate *independent* relation for the mass flux would yield precisely the equilibrium solution, with zero values for the non-equilibrium variables, for all cases that could be considered.

Experimental verification of a boundary shock wave occurring would be desirable.

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